

# Universal behavior for single-file diffusion on a disordered fractal

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**Abstract.** We study single-file diffusion on a one-dimensional lattice with a random fractal distribution of hopping rates. For finite lattices, this problem shows three clearly different regimes, namely, nearly independent particles, highly interacting particles, and saturation. The mean-square displacement of a tagged particle as a function of time follows a power law in each regime. The first crossover time  $t_s$ , between the first and the second regime, depends on the particle density. The other crossover time  $t_l$ , between the second and the third regime, depends on the lattice length. We find analytic expressions for these dependencies and show how the general behavior can be characterized by an universal form. We also show that the mean-square displacement of the center of mass presents two regimes; anomalous diffusion for times shorter than  $t_l$ , and normal diffusion for times longer than  $t_l$ .

## 1. Introduction

The subdiffusive behavior of a single random walk (RW) in a fractal medium has been explained many year ago, as the effect of obstacles of all sizes, which slow down the dynamics at every time scale [1–5]. More recently, considerable effort has been dedicated to study single-particle diffusion in finitely ramified self-similar and self-affine media. It has been repeatedly shown that, in this cases, the time behavior of the particle mean-square displacement (MSD) follows a subdiffusive power-law modulated by logarithmic-periodic oscillations [6–18].

Subdiffusion can also result from the interaction between particles, even in a substrate without any hierarchical structure. This is, for instance, the situation of a set of particles diffusing on a one-dimensional medium, in such a way that the interactions prevent for particles to jump over one another. The sequence of particles is then preserved, and the phenomenon is commonly called single-file diffusion [19–25]. In the case of single-file diffusion on an infinite homogeneous line, a representative tagged particle always starts diffusing normally, provided that interactions are of short range. However, after a finite time, the dynamics become subdiffusive, and the tagged-particle MSD behaves as  $\sim t^{1/2}$  as a function of time  $t$ , because of the correlations between neighbor particles [25]. The tagged-particle MSD recovers the normal behavior ( $\sim t$ ) at long times if single-file diffusion occurs on a homogeneous segment of finite size with periodic boundary conditions. This second crossover is closely related to existence of a growing correlation length, which becomes of the order of substrate size at the crossover time. This three-regime behavior can be expressed in a universal form, where the crossover times depend on the system size, average particle concentration, and interaction properties [24].

Single-file diffusion on a fractal has been analyzed in Ref. [26], for a set of hard-core interacting particles moving on a one-dimensional lattice with a self-similar distribution of hopping-rates [12]. In this problem, after a finite time, the tagged-particle MSD shows global subdiffusive behavior ( $\sim t^\nu$ , with  $\nu < 1/2$ ) modulated by logarithmic-periodic oscillations. The attempt to find an universal form fails because the modulations do not collapse on a single curve when scaling with respect to a single variable.

In this article, we address the problem of single-file diffusion for a set of hard-core interacting particles diffusing on a one-dimensional disordered fractal, which can be obtained by randomly shuffling the hopping rates of a self-similar lattice. As for the case of non-interacting particles [14], we show that the disorder washes the oscillations out. As a generalization of the homogeneous case, the tagged-particle MSD presents three regimes obeying a universal scaling form, which reflects the interplay between fractal properties of the substrate and hard-core interactions. We also study the center-of-mass MSD, which shows two regimes, subdiffusion at short times, and normal diffusion at long times; and is governed by another universal scaling law.

The paper is organized as follows. In section 2, we define the substrate. In section 3, we study the diffusion on non-interacting particles. Hard-core interactions,

are introduced in section 4, where we analyze how the substrate size and particle concentration affect the dynamics, and obtain universal forms for the time behavior of the MSD of a single tagged particle and that of the center of mass. Finally, we draw our conclusions in section 5.

## 2. The substrate

The substrate consists of a one-dimensional lattice, with  $M$  sites and periodical boundary conditions. The particles only jump between nearest-neighbor (NN) sites of the lattice. We express every length in units of the distance  $a$  between NN sites; which is equivalent to set  $a = 1$ . Based on the model introduced in Ref. [12], we use a discrete infinite set of hopping rates,  $q_i$  with  $i = 0, 1, 2, \dots$ , which depend on two free parameters  $L$  (integer greater than 1) and  $\lambda$  (real positive).

Starting from  $q_0$ , which fixes the time unit, the other hopping rates are defined recursively by

$$\frac{q_0}{q_i} = \frac{q_0}{q_{i-1}} + (1 + \lambda)^{i-1} \lambda L^i, \quad \text{for } i = 1, 2, 3, \dots \quad (1)$$

and are randomly distributed on the lattice with the probability distribution given by the weights

$$f_i = \frac{L - 1}{L^{i+1}}, \quad \text{for } i = 0, 1, 2, \dots \quad (2)$$

This corresponds to a fractal distribution of hopping rates, which can be considered a disordered version of the self-similar one introduced in Ref. [12].

In next sections, we address the diffusion properties of a set particles on this substrate, starting with a uniform distribution.

## 3. Non-interacting particles

We first consider the case of independent particles. We focus on the average behavior of a representative tagged particle, as expressed by its mean-square displacement  $\Delta^2 x_{\text{NI}}(t)$  at time  $t$ .

### 3.1. The infinite lattice

It has been shown that, when  $M \rightarrow \infty$ , this substrate is a disordered fractal for length scales much greater than  $L$ , and,

$$\Delta^2 x_{\text{NI}}(t) = (2D_0 t)^{2\nu} \quad (3)$$

for  $\Delta^2 x_{\text{NI}} \gg L^2$ . In equation (3),  $D_0$  is a constant, the RW exponent is given by

$$\nu = \frac{1}{2 + \frac{\log(1+\lambda)}{\log L}} \quad (4)$$

and the oscillations that modulate this overall behavior for a deterministic fractal are washed out by the random shuffling of hopping rates (for further details, see Ref. [14]).

In the rest of this article, to check our analytical results, we perform numerical Monte Carlo (MC) simulations for a substrate with  $L = 2$  and  $\lambda = 1$ . The smallest  $L$  value makes the substrate a fractal for the shortest possible distances and times. Also, from equation (4), this choice gives  $\nu = 1/3$ , which is sufficiently different from both  $\nu = 1/2$  and  $\nu = 1/4$ ; which correspond to single-particle and single-file diffusion on a homogeneous substrate, respectively. The data in figures represent averages over at least 10000 disorder realizations.

Before leaving this section, note that if we rewrite equation (3) as

$$\Delta^2 x_{\text{NI}}(t) = 2D_{\text{eff}}(t)t \quad , \quad (5)$$

the effective diffusion coefficient  $D_{\text{eff}}$  satisfies

$$2D_{\text{eff}}(t) = (2D_0)^{2\nu} t^{2\nu-1}. \quad (6)$$

Or, in terms of the characteristic RW exploration length  $\ell(t) = \sqrt{\Delta^2 x_{\text{NI}}(t)} = (2D_0 t)^\nu$ ,

$$D_{\text{eff}}(\ell) = D_0 \ell^{2-1/\nu}, \quad (7)$$

which allows us to interpret  $D_{\text{eff}}$  as a length-dependent diffusion coefficient, and subdiffusion as a result of the power-law decrease of the function  $D_{\text{eff}}(\ell)$ .

### 3.2. The finite lattice

For a lattice with a finite number of sites  $M(\gg L)$ , we expect that the tagged-particle MSD behaves as in equation (3) for length scales much shorter than  $M$  (though much larger than  $L$ ). On the other hand, for length scales much larger than  $M$ , since we work with periodic boundary conditions, the behavior is as on a homogeneous substrate. That is, we expect normal diffusion with a diffusion coefficient  $D_{\text{eff}}(\gamma M)$ , where  $\gamma$  is a number of the order of one:

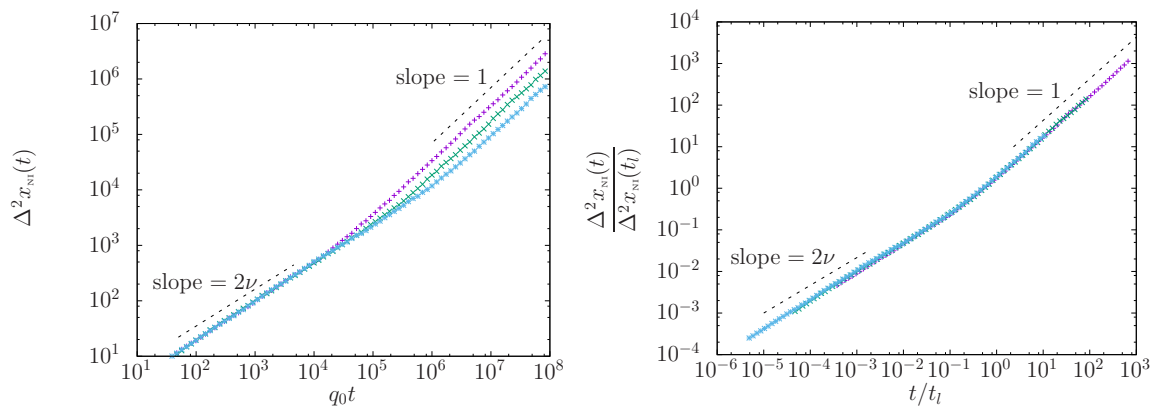
$$\Delta^2 x_{\text{NI}}(t) = 2D_{\text{eff}}(\gamma M)t \quad , \quad (8)$$

We can obtain the crossover time  $t_l$  between these two behaviors from the intersections of the right-hand sides of equations (3) and (8). Using equation (7)), we get

$$t_l = \frac{(\gamma M)^{1/\nu}}{2D_0}, \quad (9)$$

which is a generalization of the well-known result for a homogeneous substrate. Indeed, with  $\nu = 1/2$  in equation (9) we recover the behavior  $t_l \sim M^2/2D_0$  reported in Ref. [22].

In figure 1, we plot the tagged-particle MSD for various system sizes as a function of time (left panel). In the right panel of the same figure, we show the same data scaled with respect to  $t/t_l$ , using  $\gamma = 1$ . The good collapse on a single curve is consistent with the  $M$  dependence of  $t_l$  in equation (9).



**Figure 1.** (Color online) Dynamics for non-interacting particles on a disordered substrate determined by  $L = 2$  and  $\lambda = 1$ . Left: Mean-square displacement of non-interacting particles as a function of time for different lattice sizes;  $M = 50$  (violet pluses),  $M = 100$  (green crosses),  $M = 200$  (light-blue stars). Right: Scaling of the same data with respect to a single variable  $t/t_l$ . The straight lines indicate the power-law behavior in each regime, with  $\nu = 1/3$ .

#### 4. Hard-core interacting particles

If we turn on hard-core interactions between particles, the problem lies in the familiar case of single-file diffusion.

##### 4.1. Short times: the effect of particle concentration

Let us start considering an infinite substrate;  $M \rightarrow \infty$ . For short enough times, every particle behaves almost as in the non-interacting case. At these early stages of time evolution, the correlations between particle motions are negligible, and we can use a mean-field theory [25]. Within this approximation, the mean-square displacement of a tagged particle in a hard-core interacting system  $\Delta^2 x_{\text{HC}}$  satisfies

$$\Delta^2 x_{\text{HC}}(t) = (1 - c) \Delta^2 x_{\text{NI}}(t) , \quad (10)$$

where  $c$  is the average concentration of particles. Using equation (3), we can rewrite relation between the behaviors in the non-interacting and hard-core-interacting cases (10) as

$$\Delta^2 x_{\text{HC}}(t) = (1 - c)(2D_0 t)^{2\nu} . \quad (11)$$

The quantity  $(1 - c)$  represents the average probability of finding an empty site, and then, at short times, the effect of hard-core interactions is to slow down diffusion by this factor.

For longer times, after many collisions, the motions of neighbor particles become correlated and the so called single-file effect makes the diffusion even slower, which is reflected in the diminution of the RW exponent, from  $\nu$  to  $\nu/2$ . Indeed, by disordering

the hopping rates of a previously deterministic one-dimensional fractal substrate, we expect that, for long enough times, the tagged particle MSD becomes

$$\Delta^2 x_{\text{HC}} = \sqrt{\frac{2}{\pi}} \frac{(1-c)}{c} (2D_0 t)^\nu \quad (12)$$

[14, 26].

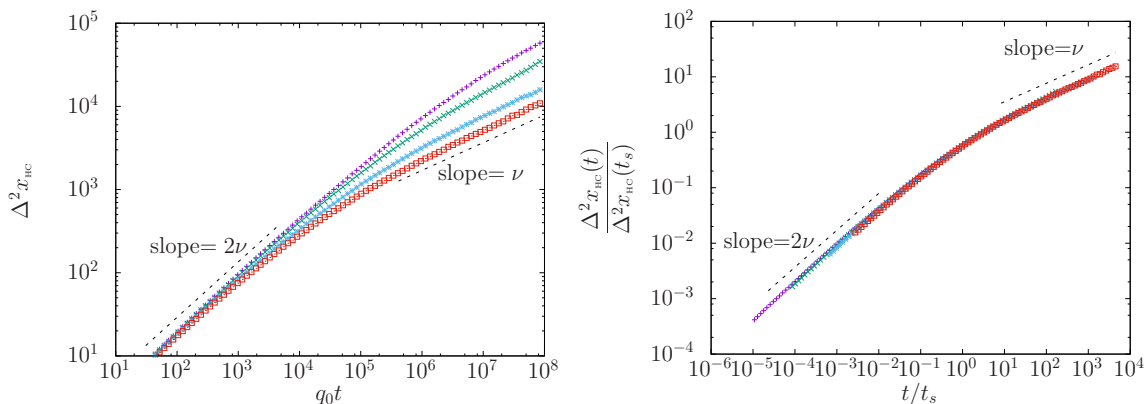
In figure 2-left, we use the results of MC simulations to plot the tagged-particle MSD as a function of time for several concentrations. The straight lines indicate the predicted short-time ( $\sim t^{2\nu}$ ) and long-time ( $\sim t^\nu$ ) behaviors, and are drawn to guide the eyes. We can also see in this figure that the crossover time between the two dynamical regimes  $t_s$  moves to the left as the concentration increases.

We calculate  $t_s$  from the intersection of the curves given by equations (11) and (12). The result is a crossover time that depends on the concentration of particles:

$$t_s = \frac{1}{2D_0} \left( \sqrt{\frac{2}{\pi}} \frac{1}{c} \right)^{1/\nu}. \quad (13)$$

This expression is a generalization of the crossover time  $t_s = 1/D_0 c^2$  that corresponds to a homogeneous substrate ( $\nu = 1/2$ ) [25].

In the right panel of figure 2, we plot in a scaled form the same data shown in left panel, i. e.,  $\Delta^2 x_{\text{HC}}(t)/\Delta^2 x_{\text{HC}}(t_s)$  against  $t/t_s$ . The good collapse on a single curve confirms the value of  $t_s$  given by equation (13).



**Figure 2.** (Color online) Short and intermediate time dynamics for hard-core interacting particles. Left: Mean-square displacement of a tagged particle for various particle concentrations;  $c = 5.0 \times 10^{-3}$  (violet pluses),  $c = 1.0 \times 10^{-2}$  (green crosses),  $c = 2.0 \times 10^{-2}$  (light-blue stars),  $c = 3.0 \times 10^{-2}$  (red squares). Right: Scaling of the same data with respect to a single variable  $t/t_s$ .

#### 4.2. Long times: the effect of substrate length

As we discussed in section 4.1, at intermediate times, the MSD of a tagged particle evolves as  $\Delta^2 x_{\text{HC}}(t) \sim t^\nu$ . It is however expected that, for even longer times, the

substrate size plays a role in the dynamics. We first investigate the effect of finite  $M$  on the time evolution of the center of mass. We will see how this analysis helps us to understand the long-time behavior of a tagged particle.

Previous studies suggest that, for an initial uniform distribution, the introduction of hard-core interactions only affects the diffusion of the center of mass by slowing it down by a factor  $(1 - c)$ . More precisely, that the relation between the center of mass MSD for non-interacting and hard-core interacting particles is

$$\Delta^2 \bar{x}_{\text{HC}} = (1 - c) \Delta^2 \bar{x}_{\text{NI}} . \quad (14)$$

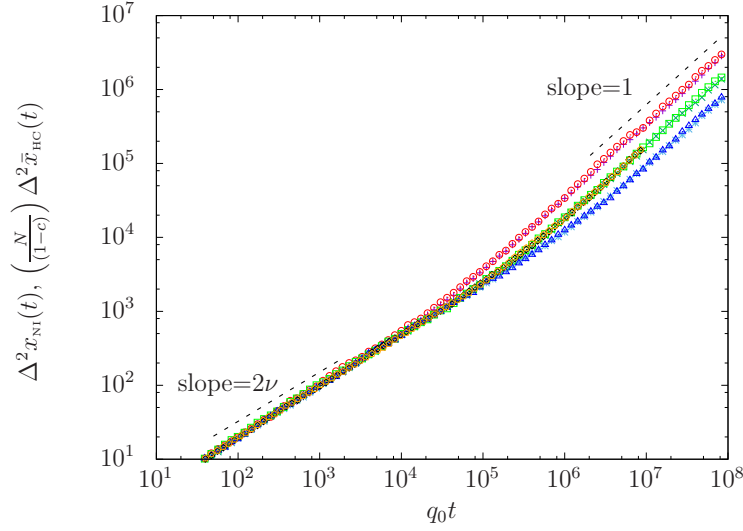
This is indeed the case of diffusion on an homogeneous lattice [27], and on a one-dimensional deterministic fractal [26] when the oscillatory modulation is averaged out. As the substrate we study here can be considered a disordered version of the last, we expect also in our case, the validity of equation (14); which, taking into account that for  $N$  non-interacting particles the relation between the MSD of a tagged particle and of a center of mass is  $\Delta^2 \bar{x}_{\text{NI}}(t) = \Delta^2 x_{\text{NI}}(t)/N$ , can be rewritten (14) as

$$\Delta^2 \bar{x}_{\text{HC}} = \frac{(1 - c)}{N} \Delta^2 x_{\text{NI}} . \quad (15)$$

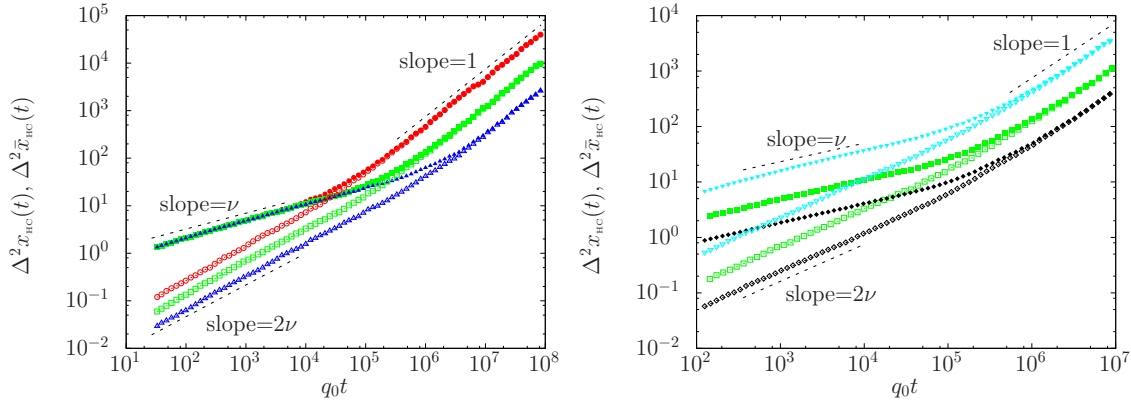
In figure 3, we show the numerical results that correspond to the time evolution of  $\Delta^2 x_{\text{NI}}$  (filled symbols) and  $(N/(1 - c))\Delta^2 x_{\text{HC}}$  (open symbols), for several concentrations and system sizes (see figure caption). The collapse of the data for every  $M$  shows that equations (14) and (15) also hold for a disordered one-dimensional fractal. As a consequence, since for a given system size and concentration the factor  $((1 - c)/N)$  in equation (15) is a constant, the center-of-mass MSD behavior crosses over from  $\sim t^{2\nu}$  to  $\sim t$  at the same time  $t_l$  as in equation (9), despite hard-core interactions.

Now, if for not short times, we compare the behavior of a tagged particle and the center of mass, we note that  $\Delta^2 \bar{x}_{\text{HC}}(t)$  ( $\sim t^{2\nu}$  or  $\sim t$ ) grows faster than  $\Delta^2 x_{\text{HC}}(t)$  ( $\sim t^\nu$ , as described by equation (12)). This cannot, however, hold forever, because the MSD of the center of mass should always be smaller or equal than the MSD of a single particle. The saturation occurs when all particles follow the same behavior.

In figure 4, we plot the MSD of both a tagged particle (filled symbols) and the center of mass (open symbols). In left panel, we use a fixed concentration  $c = 6 \times 10^{-1}$  and  $M = 50, 100, 200$ . In right panel, we use a fixed system size  $M = 100$  and  $c = 3 \times 10^{-1}, 6 \times 10^{-1}, 8 \times 10^{-1}$ . We observe that, the longer the elapsed time, the closer the values of  $\Delta^2 \bar{x}_{\text{HC}}$  and  $\Delta^2 x_{\text{HC}}$ . This is an indication of a growing correlation length, which measures the average size of a box with particles moving cooperatively. According to this, when, at time  $t_l$ , this growing length is of the order of system size  $M$ , two important changes occur. On the one hand, all particles start behaving similarly, which implies  $\Delta^2 \bar{x}_{\text{HC}}(t) \simeq \Delta^2 x_{\text{HC}}(t)$ . On the other hand, every particle has already acquired information on the whole substrate, and starts to diffuse normally as on a periodic chain;  $\Delta^2 x_{\text{HC}}(t) \sim t$ . Thus, we can use  $t_l$  also as a good estimate for the instant at which the exponent leading the time behavior of the tagged-particle MSD crosses



**Figure 3.** (Color online) Effects of interactions on evolution of the center of mass. With filled symbols, MSD of a tagged particle of non-interacting system:  $M = 50$  and  $c = 6.0 \times 10^{-1}$  (violet pluses),  $M = 100$  and  $c = 6.0 \times 10^{-1}$  (green crosses),  $M = 200$  and  $c = 6.0 \times 10^{-1}$  (light-blue stars). With open symbols, MSD of the center of mass of hard-core interacting particles, times  $N/(1 - c)$ :  $M = 50$  and  $c = 6.0 \times 10^{-1}$  (red circles),  $M = 100$  and  $c = 3.0 \times 10^{-1}$  (black diamonds),  $M = 100$  and  $c = 3.0 \times 10^{-1}$  (light-green squares),  $M = 100$  and  $c = 8.0 \times 10^{-1}$  (orange pentagons),  $M = 200$  and  $c = 6.0 \times 10^{-1}$  (blue triangles).



**Figure 4.** (Color online) Intermediate and long time dynamics of a hard-core interacting system. MSD for both a tagged particle (filled symbols) and the center of mass (open symbols) Left: Numerical results for a fixed particle concentration  $c = 6.0 \times 10^{-1}$ , and various system sizes,  $M = 50$  (red circles)  $M = 100$  (green squares),  $M = 200$  (blue upwards triangles). Right: Corresponding results for a fixed system size  $M = 100$  and various particle concentrations,  $c = 3.0 \times 10^{-1}$  (cyan downwards triangles),  $c = 6.0 \times 10^{-1}$  (green squares),  $c = 8.0 \times 10^{-1}$  (black diamonds)

over from  $\nu$  to 1. Notice that  $t_l$  does not depend on the concentration of particles, in agreement with the plots in figure 4-right.

In figure 5, we show, using the same data as in figure 4, the tagged-particle



and center-of-mass MSD's time behaviors, scaled with respect to a single variable  $t/t_l$  ( $\gamma = 1$ ). The collapse of the data on a single curve for each observable, confirms that  $t_l$  is a good estimate for the intermediate-time to long-time crossover of  $\Delta^2 x_{\text{HC}}(t)$  as it is for the crossover of  $\Delta^2 \bar{x}_{\text{HC}}(t)$ .

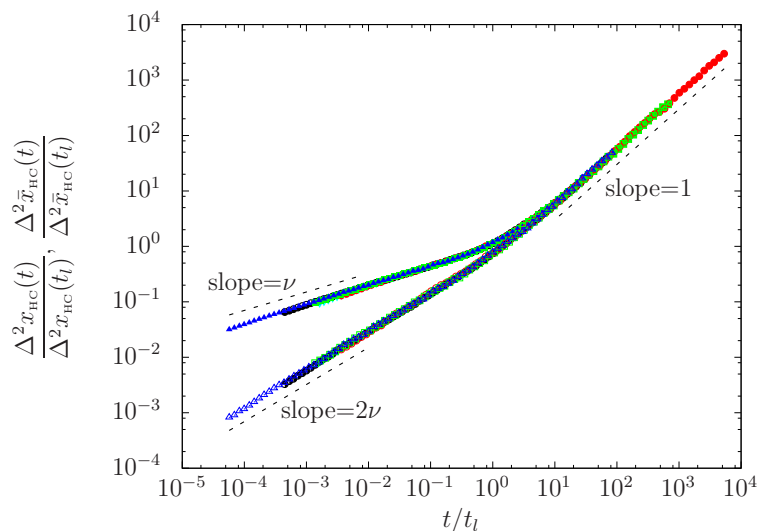
### 4.3. Universal form for the mean-square displacement

The above results allow us to state the following universal form for the MSD of the center of mass:

$$\Delta^2 \bar{x}_{\text{HC}}(t) = \Delta^2 \bar{x}_{\text{HC}}(t_l) \bar{\mathcal{G}}(t/t_l) \quad , \quad (16)$$

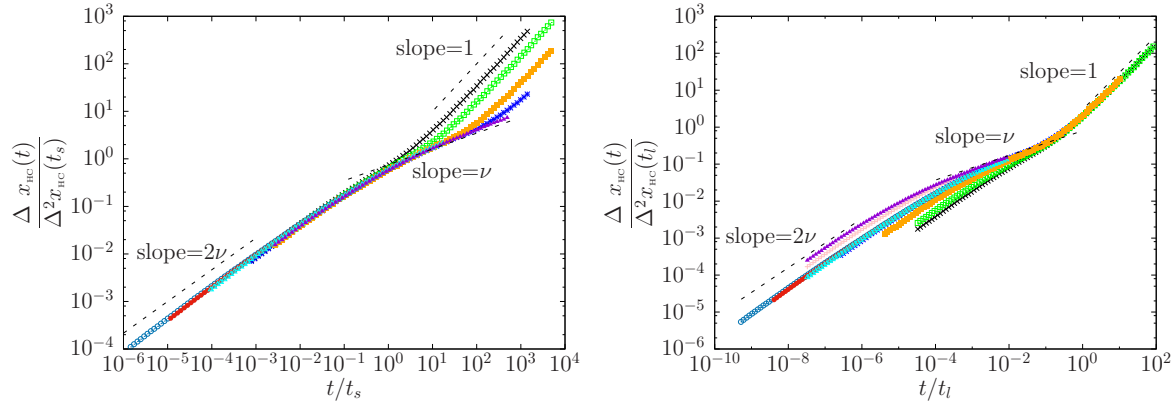
where  $\bar{\mathcal{G}}(x)$  is a universal scaling function.

The dynamics of a tagged particle is richer, and show two crossovers instead of one, at times  $t_s$  and  $t_l$ . As a consequence, it is not possible to obtain a universal scaling form, valid for every time scale, by simply scaling with respect to a single variable. For instance, the scaling of the tagged-particle MSD with respect to  $t/t_s$  leads to a collapse on a single curve for short and intermediate times ( $t \ll t_l$ ) which disappears at longer times (figure 6-left). Conversely, the scaling of the same observable with respect to  $t/t_l$  produces a good collapse of the data for long and intermediate times ( $t \gg t_s$ ), which disappears at shorter times (figure 6-right).



**Figure 5.** (Color online) Intermediate-time and long-time collapse of the MSD of both a tagged particle and the center of mass. Scaling of the same data in figure 4 with respect to a single variable  $t/t_l$ , with  $\gamma = 1$ . Symbols and colors are the same as in that figure.

For other systems, the full collapse of functions with two different crossovers has been studied in the past (see, for example, [24, 28, 29]). In the case that concerns us, the universal form of the tagged-particle MSD is achieved in two steps; a translation followed by an isotropic change of scale. First, every curve in log – log scale is rigidly



**Figure 6.** (Color online) Short, intermediate, and long time MSD of a tagged particle in a system with hard-core interactions. Left: Short-time and intermediate-time collapse of the results obtained for several particle concentrations and lattice sizes. Right: Intermediate-time and long-time collapse of the same results. Symbols in both panels:  $c = 2.0 \times 10^{-2}$  and  $M = 1000$  (pink pluses),  $c = 2.0 \times 10^{-2}$  and  $M = 100$  (black crosses),  $c = 2.0 \times 10^{-2}$  and  $M = 500$  (blue stars),  $c = 3.0 \times 10^{-2}$  and  $M = 100$  (green open squares),  $c = 3.0 \times 10^{-2}$  and  $M = 200$  (orange filled squares),  $c = 2.5 \times 10^{-3}$  and  $M = 4000$  (blue open circles),  $c = 5.0 \times 10^{-3}$  and  $M = 2000$  (red filled circles),  $c = 1.0 \times 10^{-2}$  and  $M = 1000$  (cyan open triangles),  $c = 3.0 \times 10^{-2}$  and  $M = 1000$  (violet filled triangles).

translated, to move the first (short to intermediate times) crossover point to the origin, as we do in figure 6-left, i. e., by plotting  $\left( \frac{\Delta^2 x_{\text{HC}}(t)}{\Delta^2 x_{\text{HC}}(t_s)} \right)$  as a function of  $t/t_s$ . Then, both axis are rescaled by the same factor  $1/\log\left(\frac{t_l}{t_s}\right)$ . Both operations conserve the slopes in the three regimes. The plot that results after the whole transformation, for the data in figure 6, is shown in figure 7. The very good collapse on a single curve is apparent, and gives support to the idea of universality, according to which, the MSD of a tagged particle satisfies

$$\log\left(\frac{\Delta^2 x_{\text{HC}}(t)}{\Delta^2 x_{\text{HC}}(t_s)}\right) = \log\left(\frac{t_l}{t_s}\right) \mathcal{F}\left[\frac{\log\left(\frac{t}{t_s}\right)}{\log\left(\frac{t_l}{t_s}\right)}\right], \quad (17)$$

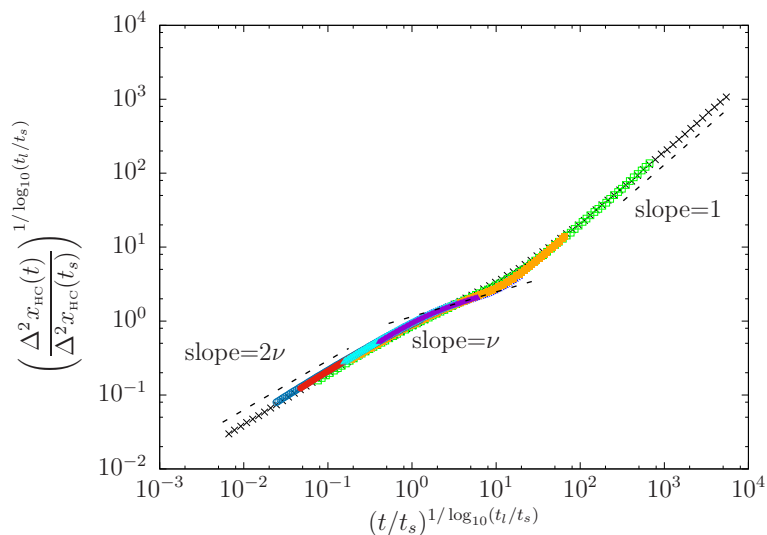
where  $\mathcal{F}(x)$  is an universal scaling function. This can also be expressed as

$$(\Delta^2 x_{\text{HC}}(t))^\mu = (\Delta^2 x_{\text{HC}}(t_s))^\mu \mathcal{G}\left[\left(\frac{t}{t_s}\right)^\mu\right], \quad (18)$$

where  $\mathcal{G}(x)$  is another scaling function, and  $\mu = 1/\log\left(\frac{t_l}{t_s}\right)$

## 5. Conclusion

In this work, we study single-file diffusion on a disordered fractal. More specifically, we focus on the MSD time behavior for the center of mass and a tagged particle. We use



**Figure 7.** Scaling form of the MSD of a tagged particle for the same data in figure 6: In this plot, we use  $t_l = \frac{M^{1/\nu}}{2D_0}$  and  $t_s = \frac{1}{2D_0} \left( \sqrt{\frac{2}{\pi}} \frac{1}{c} \right)^{1/\nu}$ . The dashed lines stand for the power-laws leading the short, intermediate, and long time behaviors (from left to right). See the main text for further details.

one-dimensional substrates of finite length, with periodic boundary conditions, where the dynamics are richer than for infinite substrates.

We propose a general relation between the center-of-mass MSD for a set of hard-core interacting particles, and the MSD of a single particle on the substrate; which we then verify numerically for all times. We can interpret that the factor  $(1 - c)/N$  between these two behaviors, comes from the fact that the system has  $N$  particles, and that the hopping rates are reduced by a factor  $(1 - c)$  due to single occupancy. We expect that the same relation will be valid for other fractal substrates.

We show that the behavior of the center-of-mass MSD presents two regimes; subdiffusion ( $\sim t^{2\nu}$ ) at short times, and normal diffusion ( $\sim t$ ) at long times. Where,  $\nu$  is the RW exponent of a single particle on the same fractal structure of infinite length. Note that for an homogeneous substrate ( $\nu = 1/2$ ), short-time and long-time exponents coincide, and the center of mass diffuses always normally, as known.

The results of the center-of-mass MSD for different lattice lengths and concentrations collapse on a single scaling form, where the crossover time, which does not depend on concentration, is the same as for non-interacting particles.

The MSD of a tagged particle presents three time regimes. In the first two, the dynamics are subdiffusive; with RW exponents  $2\nu$  and  $\nu$ , respectively. At short times no correlation between particles exists, and the tagged-particle MSD behavior is almost the same as for a single particle; only modified by the mean-field factor  $(1 - c)$ . At intermediate times, the behavior is more complex, and because of correlations, the RW exponent is reduced from  $2\nu$  to  $\nu$ . We find an analytical expression for the corresponding

crossover time, which depends on particle concentration but not on substrate length. This dependence is confirmed numerically. For long enough times, the tagged particle behaves as the center of mass, and reaches the normal diffusive regime. We predict that the crossover time between the second and third regimes has the same substrate-length dependence as the crossover time for the center-of-mass MSD, and that it does not depend on particle concentration. The outcomes of simulations confirm this prediction. The knowledge of the two crossover times allows us to formulate the universal scaling form for the tagged-particle MSD on a disordered fractal, which generalizes the one for an homogeneous substrate.

Let us finally remark that, in the infinite system size limit, the MSD of a tagged particle presents only the first two time regimes, and the dynamics of the center of mass are always subdiffusive, with RW exponent  $2\nu$ .

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